Knights and Knaves and Naive Set Theory

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Joint work with Tyler Markkanen

1996 Putnam Exam B1

*Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of* $\{1, 2, \ldots, n\}$ *which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.*

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Let $X = \{5, 6, 7, 8, 9\}$. Find a set $A \subseteq X$ with $|A| \in A$.

Paradox!

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$$A = \{2, |A|\}$$

$$B = \{1, 3, |B|\}$$

$$C = \{1, 2, 3, 4, 5, 7, |C|\}$$

Puzzle!

Notation: $|A| = a$.

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What is the cardinality of $A = \{1, 2, a, a-1\}$?

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Unique solution

What is the cardinality of $A = \{4, a, 2a\}$?

Two solutions

What is the cardinality of $A = \{1, 2, a, a-1\}$?

Three solutions

Unique Solution Cardinality Puzzles

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Consider the cases: $a = 2, 3, 4, 5$. Only one works: $A = \{3, 4, 5\}$.

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$$A = \{1, 5, 6, 10, 13, 42\}$$
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Where $f(a)$ is the line through $(6, 42)$ and $(7, 13)$

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???	$\cong$	He is a knave We are both knights

Symbiotic Sets

Let $|A| = a$ and $|B| = b$. Find the cardinalities:

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$A = \{3, b\}$	Al: Bob is a knave.
$B = \{1, a, b\}$	Bob: We are both knights.

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- Suppose $a = 1$. Then $b = 3$. But then $B = \{1, 1, 3\}$
- Suppose Al is a knave. This means Bob is a knight. But Bob's statement is false.
- Thus $a = 2$, so $B = \{1, 2, b\}$ and $b \neq 3$. So $b = 2$.
- Thus Al is a knight, so Bob is a knave (and indeed his statement is false).

What about Carl?

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$$B = \{7, 11, a, c\}$$
$$C = \{4, 7, 11, 12, 13, 14, 15, 16, a, b, c\}$$

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A set “asserts” all its elements are distinct (its size is maximal).

Open Questions

- Does every knight and knave puzzle have a matching cardinality puzzle?
- Is the correspondence better suited for multi-valued logics? There are lots of ways for a set to "lie."

Thanks!

Slides:

math.oscarlevin.com/research.php